Complex Analysis 2007.

1 (a) (i) f is differentiable at every point $z_0 \in D$.

i.e. $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists.

(ii) $f(z_0) = 0$, $f(z_0) = (z-z_0)^n g(z)$ in a neighbourhood of z_0,

g is holomorphic, $g(z_0) \neq 0$.

(b) (i) $f(z) = \frac{1}{(z^2-1)}$

(ii) $f(z) = (z-1)(z-2)(z-3)$

(iii) $\mathrm{Re}\, z = f(z)$.

(c) C-R eqns: $u_x = v_y$

$u_y = -v_x$.

<u>Proof</u>. f diff.

Let $h = h_1 + ih_2$

$$h_2 = 0 \Rightarrow f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0+h_1)-f(z_0)}{h_1} = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$$

$$h_1 = 0 \Rightarrow f'(z_0) = \lim_{h_2 \to 0} \frac{f(z_0+ih_2)-f(z_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0) = \frac{1}{i}\left[\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right]$$

$$= -i\frac{\partial u}{\partial y}(z_0) + \frac{\partial v}{\partial y}(z_0)$$

$$\Rightarrow u_x = v_y, \quad v_x = -u_y.$$

(d). At $z=0$, $f(z)=0$ so obviously satisfies CR

Examine $\lim\limits_{h\to 0} \cancel{\frac{f(z)-f(0)}{h}}$ $\cancel{f(z_0+h)}$

$h\in\mathbb{R}$ $\quad \lim\limits_{h\to 0}\frac{f(h)-f(0)}{h} = \frac{h^5}{h|h^4|} = \frac{h^4}{|h^4|} = 1$

$z = re^{i\theta} \quad \Rightarrow \quad f(z) = \frac{r^5 e^{i5\theta}}{r^4} = re^{i5\theta} = ze^{i4\theta}$

$\rightarrow$ $f'(z)$ should be $e^{i4\theta}$ for each θ. $\not\#$

On the axes, $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; $\quad e^{i4\theta} = 1$

$\rightarrow \quad f(z) = z$

?! Satisfies CRE since it cross a holomorphic f^n on the axis.

2 (a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}(-1)^n$$

$$\sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}(-1)^n$$

(b) (i) no

(ii) yes

(iii) ~~yes~~ no!

(iv) yes.

(c) Euler's formula's: $\cos z = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$\sin z = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$2\cos^2 z - 1 = 2\,\tfrac{1}{2}\left[e^{2i\theta} + 2 + e^{-2i\theta}\right] - 1 = \cos 2z.$$

(d). $z = x+iy$ $\qquad e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$$e^{2z} = e^{2x+2iy} = e^{2x}e^{2iy} = e^{2x}(\cos 2y + i\sin 2y)$$

$$e^{z^2} = e^{x^2+2xiy-y^2} = e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right)$$

$$e^{e^z} = e^{e^x(\cos y + i\sin y)} = e^{e^x(\cos y)}e^{e^x i\sin y}$$

$$= e^{e^x\cos y}\left(\cos(e^x\sin y) + i\sin(e^x\sin y)\right).$$

3. (a) The RoC of $f(z) = \sum_{n=1}^{\infty} a_n z^n$, possibly 0 or ∞

is R s.t. $\sum a_n z_n$ converges for $|z| < R$

" diverges for $|z| > R$.

(b) (i) ~~Root~~ Ratio test
$$\lim \left| \frac{(-2)^{n+1} z^{n+1}}{(n+1)^3 (-2)^n z^n} n^3 \right| = \left| (-2) z \frac{n^3}{(n+1)^3} \right|$$

$$= \left| \frac{2 z n^3}{(n+1)^3} \right| \begin{array}{l} < 1 \quad \text{if} \quad z < \frac{1}{2} \\ > 1 \quad \text{if} \quad z > \frac{1}{2}. \end{array} \qquad \underline{R = \frac{1}{2}}$$

(ii) $\cancel{\lim_{n \to \infty} \frac{(z+1)^n \, n^n}{(n+1)^{n+1} \, z^n}}$

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{z^n}{n^n} \right|} = \lim_{n \to \infty} \left| \frac{z}{n} \right| = 0 \quad \Rightarrow \quad \underline{R = \infty}.$$

(iii) $\lim_{n \to \infty} \frac{z^{2n+2}}{z^{2n}} = z^2 \qquad \underline{R = 1.}$

holomorphic when $|z| < R$.

3(c) Not on syllabus?

4 (a) Proof of Cauchy's Integral Formula.

For $f \in H(\Omega)$, Ω is simply connected,
Γ is a contour in Ω, $z_0 \in \operatorname{Int}\Gamma$.

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz .$$

Proof: Write $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz - f(z_0)$

$$= \frac{1}{2\pi i}\int \frac{f(z)-f(z_0)}{z-z_0}dz + \overbrace{\frac{1}{2\pi i}\left\{f(z_0)\int \frac{1}{z-z_0}dz\right\} - f(z_0)}^{=0}$$

$$= \frac{1}{2\pi i}\int \frac{f(z)-f(z_0)}{z-z_0}dz .$$

$F(z) = \frac{f(z)-f(z_0)}{z-z_0}$ is analytic on $\Omega\setminus\{z_0\}$

$$\Rightarrow \int_\Gamma F(z)\,dz = \int_{S(z_0,r)} F(z)\,dz ,$$

Estimate F: By def^n of $f'(z_0)$, we have

$$\forall \varepsilon>0\ \exists \delta \text{ s.t. } |z-z_0|<\delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$$

i.e. $|F(z)| < |f'(z_0)| + \varepsilon$ if $|z-z_0|<\delta$.

Take δ s.t. $\varepsilon = 1$,
then $|F(z)| < |f'(z_0)| + 1$

$$\Rightarrow \left|\int_{S(z_0,r)} F(z)\,dz\right| \leq \left[|f'(z_0)|+1\right] 2\pi r \to 0 \quad \text{as } r\to 0 .$$

$$\Rightarrow \int_{S(z_0,r)} F(z)\,dz = 0 \quad \text{as expected} \qquad \square .$$

(b) (i) $\int_\gamma \frac{z^5+i}{z-i}\,dz = 2\pi i \operatorname{Res}(f, i)$

f

$= 2\pi i \lim_{z\to i} (z^5+i) = 2\pi i (i^5+i)$

$= 2\pi i (i+i)$

$= -4\pi$

(ii) $\int_\gamma \frac{e^{z^3}}{(z-1)^3}\,dz = 2\pi i \operatorname{Res}(g, 1)$

g

$= 2\pi i \lim_{z\to 1} \frac{1}{2} \left\{ \frac{d^2}{dz^2}\left(e^{z^3}\right) \right.$

$= \cancel{\pi i \lim_{z\to 1} \left(15z^4 e^{z^3} + 9z^{10} e^{z^3}\right)}$

$\cancel{z^3 e^{z^3} 3z^2}$

$\cancel{= 3z^5 e^{z^3}}$

$\cancel{= 15z^4 e^{z^3} + 9z^{1} e^{z^3}}$

$= \pi i \left(\cancel{15e + 9e}\right)$

$\frac{d}{dz}\left(e^{z^3}\right) = e^{z^3} 3z^2$

$\frac{d^2}{dz^2}\left(e^{z^3}\right) = 9z^4 e^{z^3} + e^{z^3} 6z$

$\lim_{z\to 1} = 15e$

$= \pi i 15e$.

5 (a) (i) ~~Rem at z~~

A f^n $f(z)$ with an isolated singularity at z_0 has L. expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$.

(i) <u>Rem</u> $\forall k < 0 \quad c_k = 0$

(ii) <u>Pole</u> If $\exists M^{\in \mathbb{N}}$ s.t $\forall k < -M$, $c_k = 0$, $c_{-M} \neq 0$ pole of order M.

(iii) <u>Ess.</u> If $\nexists M \in \mathbb{Z}$ s.t. $\forall k < M$, $c_k = 0$.

(b) (i) $\frac{1}{z+\frac{1}{z}} = \frac{1}{\frac{1}{z}(1+z^2)}$ hmm . . .

⋮

(c) $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$